

The Pseudo-Newtonian Force and Potential about a Higher Dimensional Rotating Black Hole

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Abstract

In this paper, we study the behavior of the pseudo-Newtonian force and potential about a higher dimensional rotating black hole. We obtain conditions for the force character from an attractive to repulsive. We also find the conditions under which force attains a maximum value. The results of this paper generalizes the already found structure of force and potential about a five dimensional rotating black hole. It is interesting to note that we recover the five dimensional results under a special case.

Keywords: Force and Potential, Higher Dimensional Rotating Black Holes

1 Introduction

The idea of re-introducing the Newtonian gravitational force into the theory of General Relativity (GR) arose in an attempt to deal with the following problem: Gravitation, being non-linear, should dominate over the Coulomb interaction at some, sufficiently small, scale. At what scale would it occur? Whereas this question is perfectly valid in pre-relativistic terms it becomes

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meaningless in GR. The reason is that gravitation is expressed in purely geometric terms [1] while electromagnetism is not. Thus, in Relativity, gravitation possesses a very different status than the other forces of Nature. Our physical intuition for the other interactions, nevertheless, rests on the concept of forces. To deal with gravity and other forces together, we must either express the other forces geometrically, as in the Kaluza-Klein theories, or express gravitation in the same terms as the other forces. We will follow the latter alternative as the simpler program to implement.

The procedure adopted converts an idealised operational definition of the gravitational force (via the tidal force) into a mathematical formulation. In the pseudo-Newtonian (ψN) approach [2,3], the curvature of the spacetime is *straightened out* to yield a relativistic force which bends the path, so as to again supply the guidance of the earlier, force-based, intuition. Some insights have already been obtained [2-5] by expressing the consequences of GR in terms of forces by applying it to Kerr and Kerr-Newmann metrics. The ψN potential has also been evaluated for charged particle in Kerr-Newmann geometry by Ivanov and Prodanov [6]. In a recent paper [7], the structure of the ψN force and potential has been analyzed about a five dimensional rotating black hole and some insights have been achieved. This paper investigates the structure of the ψN force and potential about a higher dimensional rotating black hole.

The paper is organized as follows: In the next section, we briefly discuss the ψN formalism so that the definition of force and potential can be given in this formalism. Section 3 provides the details of the metric describing higher dimensional rotating black hole. We calculate the force and potential for this metric in section 4. Finally, section 5 concludes the results.

2 The Pseudo-Newtonain Formalism

The basis of the formalism is the observation that the tidal force, which is operationally determinable, can be related to the curvature tensor by

$$F_T^\mu = m R_{\nu\rho\pi}^\mu t^\nu l^\rho t^\pi, \quad (\mu, \nu, \rho, \pi = 0, 1, 2, 3), \quad (1)$$

where m is the mass of a test particle, $t^\mu = f(x)\delta_0^\mu$, $f(x) = (g_{00})^{-1/2}$ and l^μ is the separation vector. l^μ can be determined by the requirement that the tidal force have maximum magnitude in the direction of the separation vector. Choosing a gauge in which $g_{0i} = 0$ (similar to the synchronous coordinate

system [8]) in a coordinate basis. We further use Riemann normal coordinates (RNCs) for the spatial direction, but not for the temporal direction. The reason for this difference is that both ends of the accelerometer are spatially free, i.e., both move and do not stay attached to any spatial point. However, there is a *memory* of the initial time built into the accelerometer in that the zero position is fixed then. Any change is registered that way. Thus *time* behaves very differently from *space*.

The relativistic analogue of the Newtonian gravitational force called the ψN gravitational force, is defined as the quantity whose directional derivative along the accelerometer, placed along the principal direction, gives the extremised tidal force and which is zero in the Minkowski space. Thus the ψN force, F^μ , satisfies the equation

$$F_T^{*\mu} = l^\nu F_{;\nu}^\mu, \quad (2)$$

where $F_T^{*\mu}$ is the extremal tidal force corresponding to the maximum magnitude reading on the dial. Notice that $F_T^{*0} = 0$ does not imply that $F^0 = 0$. With the appropriate gauge choice and using RNCs spatially, Eq.(2) can be written in a space and time break up as

$$l^i(F_{,i}^0 + \Gamma_{ij}^0 F^j) = 0, \quad (3)$$

$$l^j(F_{,j}^i + \Gamma_{0j}^i F^0) = F_T^{*i}, \quad (i, j = 1, 2, 3). \quad (4)$$

A simultaneous solution of the above equations can be found by taking the ansatz [3]

$$F_0 = -m \left[\{\ln(Af)\}_{,0} + g^{ik} g_{jk,0} g^{jl} g_{il,0} / 4A \right], \quad (5)$$

$$F_i = m(\ln f)_{,i}, \quad (6)$$

where $A = (\ln \sqrt{-g})_{,0}$, $g = \det(g_{ij})$. It is mentioned here that this force formula depends on the choice of frame, which is not uniquely fixed.

The spatial component of the ψN force F_i is the generalisation of the force which gives the usual Newtonian force for the Schwarzschild metric and a $\frac{Q^2}{r^3}$ correction to it in the Riessner-Nordstrom metric. The ψN force may be regarded as the *Newtonian fiction* which *explains* the same motion (geodesic) as the *Einsteinian reality* of the curved spacetime does. We can, thus, translate back to Newtonian terms and concepts where our intuition may be able to lead us to ask, and answer, questions that may not have

occurred to us in relativistic terms. Notice that F_i does not mean deviation from geodesic motion.

The quantity whose proper time derivative is F_μ gives the momentum four-vector for the test particle. Thus the momentum four-vector, p_μ , is [4]

$$p_\mu = \int F_\mu dt. \quad (7)$$

The zero component of the momentum four-vector corresponds to the energy imparted to a test particle of mass m while the spatial components of this vector give the momentum imparted to test particles as defined in the preferred frame (in which $g_{0i} = 0$). We can also write Eqs.(5) and (6) as follows

$$F_0 = -U_{,0}, \quad F_i = -V_{,i}, \quad (8)$$

where the quantities U and V are given by

$$U = m[\ln(Af/B) + \int (g_{,0}^{ij} g_{ij,0}/4A) dt], \quad (9)$$

$$V = -m \ln f. \quad (10)$$

Here B is a constant with units of time inverse so as to make A/B dimensionless.

In the free fall rest-frame, the ψN force is given [3,4] by

$$F_i = -m(\ln \sqrt{g_{00}})_{,i} = -V_{,i}, \quad (11)$$

where

$$V = m(\ln \sqrt{g_{00}}). \quad (12)$$

This quantity V gives the generalization of the classical gravitational potential and, for small variations from Minkowski space

$$V \approx \frac{1}{2}m(g_{00} - 1) \quad (13)$$

which is the pseudo-Newtonian potential. We shall analyse the behavior of these quantities for the higher dimensional rotating black hole in the next section.

3 Higher Dimensional Rotating Black Hole

The metric of a rotating black hole in higher dimensions follows from the general asymptotically flat solutions to $(N + 1)$ dimensional vacuum gravity found by Myers and Perry [9]. We consider higher dimensional rotating black hole with a single rotation parameter. In Boyer-Lindquist type coordinates, the metric is given by [10]

$$\begin{aligned} ds^2 = & \left(1 - \frac{M}{r^{N-4}\Sigma}\right)dt^2 - \frac{r^{N-2}\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 \\ & - \left(r^2 + a^2 + \frac{Ma^2 \sin^2 \theta}{r^{N-4}\Sigma}\right) \sin^2 \theta d\phi^2 \\ & + \frac{2Ma \sin^2 \theta}{r^{N-4}\Sigma} dt d\phi - r^2 \cos^2 \theta d\Omega_{N-3}^2, \end{aligned} \quad (14)$$

where

$$\Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^{N-2}(r^2 + a^2) - Mr^2, \quad (15)$$

and M is a parameter related to the physical mass of the black hole, while the parameter a is associated with its angular momentum. The quantity

$$d\Omega_{N-3}^2 = d\chi_1^2 + \sin^2 \chi_1 (d\chi_2^2 + \sin^2 \chi_2 (\dots d\chi_{N-3}^2 \dots)) \quad (16)$$

represents the metric of a unit $(N - 3)$ -sphere.

It is mentioned here that for $N = 3$, neglecting the last term of Eq.(14), this gives the analogue of the Kerr metric. Also, for $N = 4$, this exactly reduces to the metric representing five dimensional rotating black hole with $b = 0$ [7]. The event horizon of the black hole is a null surface determined by the equation

$$\Delta = r^2 + a^2 - \frac{M}{r^{N-4}} = 0. \quad (17)$$

The largest root of this equation gives the radius of the black hole's outer event horizon. Notice that for $N = 3$ and $N = 4$, the horizon exists unless its rotation achieves the maximum speed by the mass of the black hole. For $N \geq 5$, the horizon exists independent of the rotation [9,11]. The rotational symmetry in the ϕ -direction along with the time-translation invariance of the metric imply the existence of the commuting Killing vectors

$$\xi_{(0)} = \xi_{(t)}^\mu \frac{\partial}{\partial x^\mu}, \quad \xi_{(3)} = \xi_{(\phi)}^\mu \frac{\partial}{\partial x^\mu}. \quad (18)$$

The scalar products of these Killing vectors can be written down in terms of the metric components as given below:

$$\begin{aligned}\xi_{(0)} \cdot \xi_{(0)} &= g_{00} = 1 - \frac{M}{r^{N-4}\Sigma}, \\ \xi_{(0)} \cdot \xi_{(3)} &= g_{03} = \frac{Ma \sin^2 \theta}{r^{N-4}\Sigma}, \\ \xi_{(3)} \cdot \xi_{(3)} &= g_{33} = -(r^2 + a^2 + \frac{Ma^2 \sin^2 \theta}{r^{N-4}\Sigma}) \sin^2 \theta.\end{aligned}\quad (19)$$

The Killing vectors (18) can be used to give a physical interpretation of the parameters M and a involved in the metric (14). The following coordinate-independent definitions for these parameters can be obtained by using the analysis given in the paper [12]. Thus

$$M = \frac{1}{(N-2)A_{N-1}} \oint \xi_{(t)}^{\mu;\nu} d^{N-1}\Sigma_{\mu\nu} \quad (20)$$

and

$$j = aM = -\frac{1}{4\pi^2} \oint \xi_{(\phi)}^{\mu;\nu} d^{N-1}\Sigma_{\mu\nu}. \quad (21)$$

The integrals are taken over the $(N-1)$ -sphere at spatial infinity

$$d^{N-1}\Sigma_{\mu\nu} = \frac{1}{(N-1)!} \sqrt{-g} \epsilon_{\mu\nu i_1 i_2 \dots i_{N-1}} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_{N-1}} \quad (22)$$

and

$$A_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)} \quad (23)$$

is the area of a unit $(N-1)$ -sphere. These definitions can be verified by evaluating the integrands in the asymptotic region $r \rightarrow \infty$. The dominant terms in the asymptotic expansion take the following form

$$\begin{aligned}\xi_{(t)}^{t;r} &= \frac{M(N-2)}{2r^{N-1}} + O\left(\frac{1}{r^{N+1}}\right), \\ \xi_{(\phi)}^{t;r} &= -\frac{jN \sin^2 \theta}{2r^{N-1}} + O\left(\frac{1}{r^{N+1}}\right).\end{aligned}\quad (24)$$

It is obvious that these expressions justify Eqs.(20) and (21). The total mass M_T and the total angular momentum J of the black hole can be found [9] as

$$M = \frac{16\pi G M_T}{(N-1)A_{N-1}}, \quad j = \frac{8\pi G J}{A_{N-1}}. \quad (25)$$

This equation justifies the interpretation of the parameters M and a related to the physical mass and angular momentum of the black hole.

4 The Pseudo-Newtonian Force and Potential

In this section, we calculate ψN force and potential for the higher dimensional rotating black hole and analyze them. Using Eqs.(11) and (14), the structure of the ψN force (per unit mass of the test particle) for the higher dimensional rotating black hole takes the following form

$$F_r = -\frac{M[(N-4)r^{N-3}\Sigma + 2r^{N-3}]}{2r^{N-4}\Sigma(r^{N-4}\Sigma - M)}, \quad (26)$$

$$F_\theta = \frac{Ma^2 \sin \theta \cos \theta}{r^{N-4}\Sigma(r^{N-4}\Sigma - M)}. \quad (27)$$

It follows from Eq.(26) that the radial component can never become zero outside the horizon and hence this force cannot change character from an attractive to a repulsive outside the black hole. From Eq.(27), we see that the polar component can only become zero outside the horizon at $\theta = 0, \pi/2, \pi$. Notice that naked singularities can give repulsive as well as attractive forces. When the turnover lies outside the horizon, the structure of force can provide interesting features for its optimal values according to r or θ . Since our observers are seeing *force* in a flat space, the metric to be used is the plane polar one. The square of the magnitude of the force is

$$(F)^2 = \frac{M^2}{4r^{2N-6}\Sigma^2(r^{N-4}\Sigma - M)^2} [r^{2N-4}((N-4)\Sigma+2)^2 + 4a^4 \sin^2 \theta \cos^2 \theta] \quad (28)$$

which implies that

$$|F| = \frac{M}{2r^{N-3}\Sigma(r^{N-4}\Sigma - M)} [r^{2N-4}((N-4)\Sigma+2)^2 + 4a^4 \sin^2 \theta \cos^2 \theta]^{1/2}. \quad (29)$$

When we expand it in powers of $(1/r)$, it follows that

$$|F| = \frac{Mr((N-4)\Sigma+2)}{2\Sigma(r^{N-4}\Sigma - M)} \left[1 + \frac{2a^4}{r^{2N-4}((N-4)\Sigma+2)^2} \sin^2 \theta \cos^2 \theta + \dots \right]. \quad (30)$$

The equations for the turnovers along r and θ , respectively, are

$$\begin{aligned}
& r^{2N-4}\Sigma(r^{N-4}\Sigma - M)((N-4)\Sigma + 2)[(2N-4)((N-4)\Sigma + 2) \\
& + 4(N-4)] - [r^{2N-4}((N-4)\Sigma + 2)^2 + 4a^4 \sin^2 \theta \cos^2 \theta] \\
& \times [(2N-6)\Sigma(r^{N-4}\Sigma - M) + 4r^2(r^{N-4}\Sigma - M) \\
& + 2r^{N-2}\Sigma((N-4)\Sigma + 2)] = 0,
\end{aligned} \tag{31}$$

$$\begin{aligned}
& 2\Sigma(r^{N-4}\Sigma - M)[a^2 \cos 2\theta - r^{2N-4}(N-4)((N-4)\Sigma + 2)] \\
& + (r^{N-4}\Sigma - M + \Sigma)[4a^4 \sin^2 \theta \cos^2 \theta + r^{2N-4}((N-4)\Sigma + 2)^2] = 0
\end{aligned} \tag{32}$$

The complexity of Eqs.(31) and (32) makes it difficult to analyse them generally. However, we can investigate these equations for the following two special cases.

- (i) $N = 4, \quad a \neq 0, \quad$ (ii) $N = 4, \quad a = 0.$

The first case (i) exactly reduces to the results of the paper [7]. In this case, it follows from Eqs.(31) and (32) that

$$(r^2 + a^2)(r^2 + a^2 - M) - 2r^2(2r^2 + 2a^2 - M) = 0, \tag{33}$$

$$2(r^2 + a^2) - M = 0. \tag{34}$$

The first equation is satisfied when

$$r^2 = \frac{1}{6}[(M - 2a^2) \pm \sqrt{M^2 + 16a^4 - 16Ma^2}]. \tag{35}$$

Eq.(34) is satisfied for the value of r given by

$$r^2 = \frac{1}{2}M - a^2. \tag{36}$$

For $N = 4, \quad a = 0$, i.e., when there is no rotation, Eqs.(31) and (32), respectively, are satisfied for the values of r given as

$$r = \pm \sqrt{\frac{M}{3}}, \quad r = \pm \sqrt{\frac{M}{2}}. \tag{37}$$

In the first case (i), a maximum of the magnitude can be achieved at the value of r given by Eq.(35) and hence the maximum value of the force can

be obtained by replacing this value of r in F_r . For the case (ii), a maximum value of the magnitude occurs at r as given in Eq.(37).

The corresponding potential is obtained by using Eqs.(13) and (14) and is given by

$$V = -\frac{M}{2r^{N-4}\Sigma}. \quad (38)$$

For the special case (i), it becomes

$$V = -\frac{M}{2(r^2 + a^2)}. \quad (39)$$

When there is no rotation, it reduces to

$$V = -\frac{M}{2r^2}. \quad (40)$$

It is remarked that the structure of force and potential indicate similar type of behavior as for the Kerr metric [4] in the special cases.

5 Summary

The relativistic analogue of the Newtonian gravitational potential is related to the square of the magnitude of the timelike Killing vector [13]. This conjecture is verified as an approximation and has been applied to the Schwarzschild metric. Later, its validity was justified for a particular class of spacetimes [5]. Thus the procedure of looking for the implications of relativity in terms of an analogue of the Newtonian gravitational force [4] has already provided some insights [1-5,14].

This paper is particularly emphasized to analyse the structure of the pseudo-Newtonian force and potential about a higher dimensional rotating black hole. When we take $N = 4$, this gives the same results as given in the paper [6] with only one angular momentum. We have found that the radial component of the force cannot change character from an attractive to a repulsive outside the black hole as it never becomes zero there. The polar component becomes zero outside the horizon at $\theta = 0, \pi/2, \pi$. When the turnover lies outside the horizon, the structure of the force can provide interesting features for its optimal values according to r or θ . The equations for the turnovers along r and θ are too difficult to analyse. However, we have investigated these equations for the two special cases, i.e., $N = 4, a \neq 0$

and $N = 4$, $a = 0$. The first case exactly reduces to the results of the paper [7] while the second case corresponds to no rotation. It is worth mentioning that the general case gives similar type of behavior of the force and potential as for the Kerr metric [5] in the special cases.

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